Determinant Line Bundles Revisited

Daniel S. Freed

Department of Mathematics
University of Texas at Austin

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This note is an addendum to joint work with Xianzhe Dai [DF1], [DF2].[1] In that paper we investigate the geometric theory of η-*invariants* of Dirac operators on manifolds with boundary. We summarize the main results below. One key geometric observation is that the exponentiated η-invariant naturally takes values in the *determinant line* of the boundary. As such it is intimately related to the geometry of determinant line bundles for *families* of Dirac operators. The differential geometry of determinant line bundles was developed first by Quillen [Q] in a special case, and then by Bismut and Freed [BF1], [BF2] in general. (See [F1] for an exposition of these results.) In §5 of [DF1] the results on η-invariants are used to reprove the holonomy formula for determinant line bundles, also known as Witten's global anomaly formula [W]. However, the argument there is unnecessarily complicated. The main purpose of this note, then, is to reprove *both* the curvature and holonomy formulas for determinant line bundles using the results of [DF1]. (The argument was sketched in [DF2].)

To avoid repetitious recitation of requirements, we set some conventions here which apply throughout. We work with *compact* Riemannian manifolds. If the boundary is nonempty we assume that the metric is a product near the boundary. Our results hold for any Dirac operator on a spin^c manifold coupled to a vector bundle with connection, but for simplicity we state the formulas only for the basic Dirac operator on a spin manifold. Thus all manifolds are assumed spin. We use the L^2 metric on the spinor fields S. A *family of Riemannian manifolds* is a smooth fiber bundle $\pi\colon X \to Z$ together with a metric on the relative (vertical) tangent bundle $T(X/Z)$ and a distribution of "horizontal" complements to $T(X/Z)$ in TX. We assume that $T(X/Z)$ is endowed with a spin structure. Also, when working with families of manifolds with boundary, we assume that the Riemannian metrics on the fibers are products near the boundary. There is an

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[1] In [DF1] the reader will find an extensive discussion of related work and a bibliography.

induced family $\partial\pi\colon \partial X \to Z$ of closed manifolds. Finally, we will always use 'X' to denote an odd dimensional manifold and 'Y' to denote an even dimensional manifold.

As stated earlier this is a continuation of joint work with Xianzhe Dai.

Eta Invariants on Manifolds with Boundary

First recall that on a *closed* odd dimensional manifold X the Dirac operator D_X is self-adjoint and has a discrete spectrum $\operatorname{spec}(D_X)$ extending to $+\infty$ and $-\infty$. The η-invariant of Atiyah-Patodi-Singer [APS] is defined by meromorphic continuation of the function

$$\eta_X(s) = \sum_{\substack{\lambda\neq 0\\ \lambda\in\operatorname{spec}(D_X)}} \frac{\operatorname{sign}\lambda}{|\lambda|^s},$$

which by general estimates converges for $\operatorname{Re}(s)$ sufficiently large. In fact, for Dirac operators the meromorphic continuation is analytic for $\operatorname{Re}(s) > -2$ [BF2, Theorem 2.6]. In any case η_X is regular at $s = 0$, and we set

$$\tau_X = \exp \pi i(\eta_X(0) + \dim\operatorname{Ker} D_X) \in \mathbb{C}. \tag{1}$$

The general theory of η-invariants shows that τ_X varies smoothly in families, whereas the η-invariant $\eta_X(0)$ is discontinuous in general. Note that $|\tau_X| = 1$.

On a manifold with boundary we need to specify elliptic boundary conditions to obtain an operator with discrete spectrum. We use the boundary conditions introduced by Atiyah-Patodi-Singer, but adapted to odd dimensional manifolds X. This involves an additional piece of information concerning $\operatorname{Ker} D_{\partial X}$. Recall that on an even dimensional manifold Y the spinor fields S_Y split as $S_Y = S_Y^+ \oplus S_Y^-$, and the Dirac operator $D_Y\colon S_Y^{\pm} \to S_Y^{\mp}$ interchanges the positive and negative pieces. (In the sequel we use 'D_Y' to denote the operator $D_Y\colon S_Y^+ \to S_Y^-$.) If $Y = \partial X$ is the boundary of an odd dimensional manifold X, then $\dim\operatorname{Ker}^+ D_{\partial X} = \dim\operatorname{Ker}^- D_{\partial X}$. The additional piece of information we must choose as part of the boundary condition is an isometry

$$T\colon \operatorname{Ker}^+ D_{\partial X} \longrightarrow \operatorname{Ker}^- D_{\partial X}.$$

Then the basic analytic properties of D_X with these boundary conditions are the same as those of the Dirac operator on a closed manifold, and so the invariant (1) is defined. Its dependence on T is simple, and factoring this out we observe that

$$\tau_X \in \operatorname{Det}^{-1}_{\partial X}, \tag{2}$$

where $\mathrm{Det}_{\partial X}$ is the *determinant line* of the Dirac operator $D_{\partial X}$ on the boundary:

$$\mathrm{Det}_{\partial X} = \big(\mathrm{Det}\,\mathrm{Ker}^-\,D_{\partial X}\big) \otimes \big(\mathrm{Det}\,\mathrm{Ker}^+\,D_{\partial X}\big)^{-1}. \tag{3}$$

(Recall that $\mathrm{Det}\,V = \bigwedge^n V$ for an n dimensional vector space V. Also $L^{-1} = L^*$ for a one dimensional vector space L.) Properly normalized we have $|\tau_X| = 1$ in the *Quillen metric* on $\mathrm{Det}_{\partial X}^{-1}$.

Now suppose $X \to Z$ is a family of odd dimensional manifolds with boundary. Then $\partial X \to Z$ is a family of closed even dimensional manifolds. The determinant lines (3) patch together to form a smooth determinant line bundle $\mathrm{Det}_{\partial X/Z} \to Z$. Furthermore, it carries the Quillen metric and a canonical connection ∇, as defined in [BF1]. The exponentiated η-invariant is now a smooth section

$$\tau_{X/Z}\colon Z \longrightarrow \mathrm{Det}_{\partial X/Z}^{-1}.$$

There are two basic results about this invariant: a variation formula and a gluing law. The variation formula computes the derivative of $\tau_{X/Z}$ in a family.

Theorem 4 [DF1,Theorem 1.9]**.** *With respect to the canonical connection* ∇ *on* $\mathrm{Det}_{\partial X/Z}^{-1}$,

$$\nabla \tau_{X/Z} = 2\pi i \left[\int_{X/Z} \hat{A}(\Omega^{X/Z})\right]_{(1)} \cdot \tau_{X/Z}.$$

Here $\Omega^{X/Z}$ is the Riemannian curvature of $X \to Z$ and $\hat{A}$ is the usual $\hat{A}$-polynomial. (For other Dirac operators substitute the appropriate index polynomial in place of $\hat{A}$.) The '(1)' denotes the 1-form piece of the differential form. For a family of *closed* manifolds this is a result of Atiyah-Patodi-Singer. The new point here is the relationship of τ with the canonical connection ∇. This plays a crucial role in the next section.

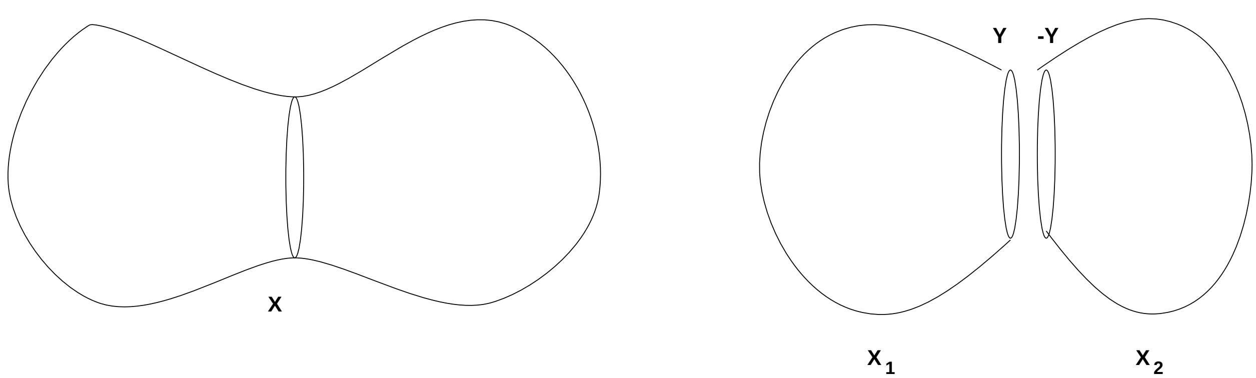

FIGURE 1. Cutting a closed manifold into two pieces.

The simplest case of the gluing law is for a *closed* manifold X split into two pieces X_1, X_2 along a closed oriented codimension one submanifold $Y \hookrightarrow X$. (See Figure 1.) Then $\tau_{X_i} \in \mathrm{Det}_Y^{-1}$ and $\tau_X \in \mathbb{C}$.

Theorem 5 [DF1,Theorem 2.20]**.** *In this situation*

$$\tau_X = (\tau_{X_1}, \tau_{X_2})_{\mathrm{Det}_Y^{-1}}.$$

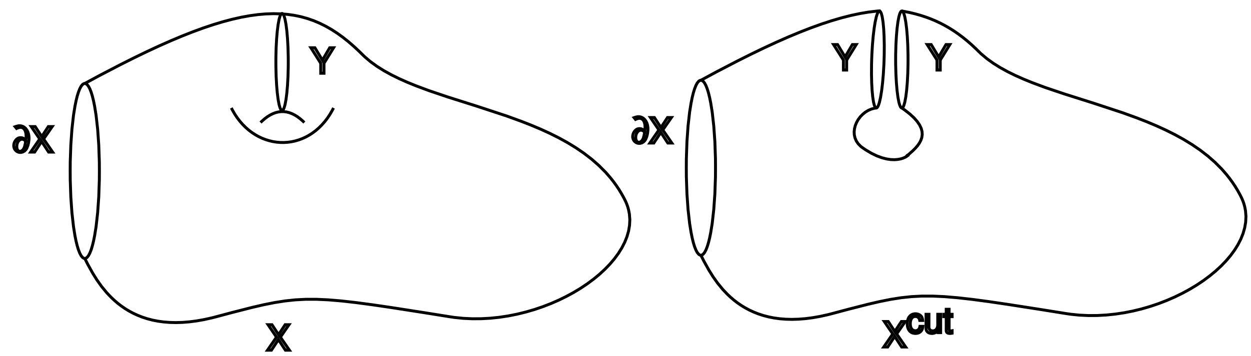

FIGURE 2. Cutting a manifold along a submanifold.

The more general gluing formula, which we need in the next section, applies when X has boundary. Then for $Y \hookrightarrow X$ a closed oriented codimension one submanifold we cut along Y to obtain a new manifold X^{cut} with $\partial X^{\mathrm{cut}} = \partial X \sqcup Y \sqcup -Y$. (See Figure 2.) Now

$$\begin{aligned} \tau_X &\in \mathrm{Det}_{\partial X}^{-1} \\ \tau_{X^{\mathrm{cut}}} &\in \mathrm{Det}_{\partial X}^{-1} \otimes \mathrm{Det}_Y^{-1} \otimes \mathrm{Det}_{-Y}^{-1} \\ &\cong \mathrm{Det}_{\partial X}^{-1} \otimes \ L_Y \ \otimes \ L_Y^{-1}, \end{aligned} \tag{6}$$

where $L_Y = \mathrm{Det}_Y^{-1}$. There is now a sign which enters the gluing formula, and it is nicely taken care of by the following device. In general we view the determinant line $\mathrm{Det}\, V$ of a vector space V as a one dimensional *graded* vector space whose grading is given by $\dim V$. Applied to (3) we see that Det_Y (and so also Det_Y^{-1}) is graded by the *index* of the Dirac operator D_Y. Notice that in our current situation Y does not necessarily bound a 3-manifold, and so its index may be nonzero. Let

$$\mathrm{Tr}_s \colon L_Y \otimes L_Y^{-1} \longrightarrow \mathbb{C} \tag{7}$$

be the usual contraction times the grading; i.e., if $\mathrm{index}\, D_Y$ is even it is the usual contraction and if $\mathrm{index}\, D_Y$ is odd it is minus the usual contraction. That understood, we state the general gluing formula.

Theorem 8 [DF1, Theorem 2.20]**.** *In this situation*

$$\tau_X = \mathrm{Tr}_s(\tau_{X^{\mathrm{cut}}}). \tag{9}$$

One of the novel points of [DF1] is the proof of the gluing law, which we do not discuss here.

Determinant Line Bundles and Adiabatic Limits

The application we discuss is to the geometry of the determinant line bundle. Suppose $\pi\colon Y \to Z$ is a family of *closed* even dimensional manifolds. Let $L = \mathrm{Det}^{-1}_{Y/Z}$ be the inverse determinant line bundle of the family. The results in the last section use the Quillen metric and the construction of the canonical connection ∇. But they do not depend on the formulas for the curvature and holonomy of ∇, which were proved in [BF1], [BF2]. Here we derive the curvature and holonomy formulas from Theorem 4 and Theorem 8.[2] The basic idea is to use the τ-invariant (2) to define the parallel transport of a new connection ∇' on L. Thus suppose $\gamma\colon [0,1] \to Z$ is a smooth path[3] in Z. Denote $I = [0,1]$. Let $Y_\gamma = \gamma^*(Z) \to I$ be the pullback of the family $\pi\colon Y \to Z$ by the path γ. Then Y_γ is an odd dimensional manifold with $\partial Y_\gamma = Z_{\gamma(1)} \sqcup -Z_{\gamma(0)}$. The standard metric g_I on $I = [0,1]$ determines a metric on Y_γ, since we already have a metric $g_{Y_\gamma/I}$ on the fibers and a distribution of horizontal planes. (The projection $\pi\colon Y_\gamma \to I$ is then a Riemannian submersion.) The τ-invariant of Y_γ is a linear map

$$\tau_{Y_\gamma}\colon L_{\gamma(0)} \longrightarrow L_{\gamma(1)}, \tag{10}$$

exactly what we need to define parallel transport. However, (10) does *not* define parallel transport since it is not independent of the parametrization of the path γ. To get a quantity independent of parametrization we introduce the *adiabatic limit* as follows. For each $\epsilon \neq 0$ consider the metric

$$g_\epsilon = \frac{g_I}{\epsilon^2} \oplus g_{Y_\gamma/I} \tag{11}$$

on Y_γ relative to the decomposition $TY_\gamma \cong \pi^*TI \oplus T(Y_\gamma/I)$. Let $\tau_{Y_\gamma}(\epsilon)$ be the τ-invariant for this metric.

Lemma 12. *The adiabatic limit*

$$\tau_\gamma = \text{a-lim}\,\tau_{Y_\gamma} = \lim_{\epsilon\to 0}\tau_{Y_\gamma}(\epsilon) \tag{13}$$

[2] As was mentioned in the introduction, this was done in [DF1,§5] in an unnecessarily complicated way. Also, there we *used* the curvature formula instead of *proving* it. This section should be considered a rewrite of [DF1,§5].

[3] Since we need a cylindrical metric near the boundary of Y_γ defined below, we require that $\gamma([0,\delta])$ and $\gamma([1-\delta,1])$ be constant for some δ.

exists and is invariant under reparametrization of γ*.*

Notice that the adiabatic limit is introduced for a simple geometrical reason—to scale out the dependence of τ on the parametrization of γ.

Proof. Here we follow [DF1,§5].[4] As a preliminary we state without proof a simple result about the Riemannian geometry of adiabatic limits. Let $\nabla^{Y_\gamma}(\epsilon)$ denote the Levi-Civita connection on Y_γ of the metric (11) and $\Omega^{Y_\gamma}(\epsilon)$ its curvature. The result we need, which follows from a straightforward computation in local Riemannian geometry, is that $\text{a-lim}\,\nabla^{Y_\gamma} = \lim\limits_{\epsilon\to 0}\nabla^{Y_\gamma}(\epsilon)$ exists and is torsionfree. Furthermore, the curvature of this limiting connection is the limit of the curvatures of $\nabla^{Y_\gamma}(\epsilon)$ and has the form

$$\text{a-lim}\,\Omega^{Y_\gamma} = \lim_{\epsilon\to 0}\Omega^{Y_\gamma}(\epsilon) = \begin{pmatrix} 0 & * \\ 0 & \Omega^{Y_\gamma/I}\end{pmatrix}$$

relative to a fixed (nonorthonormal) basis. It follows that

$$\text{a-lim}\,\hat{A}(\Omega^{Y_\gamma}) = \lim_{\epsilon\to 0}\hat{A}(\Omega^{Y_\gamma}) = \hat{A}(\Omega^{Y_\gamma/I}). \tag{14}$$

We apply this result to families of adiabatic limits, where it also holds.

To prove that the adiabatic limit exists, consider the family of Riemannian manifolds $Y_\gamma\times\mathbb{R}^{\neq 0}\to\mathbb{R}^{\neq 0}$, where the metric on the fiber at ϵ is (11). According to the variation formula Theorem 4 we have

$$\frac{d}{d\epsilon}\tau_{Y_\gamma}(\epsilon) = 2\pi i\left[\int_{(Y_\gamma\times\mathbb{R}^{\neq 0})/\mathbb{R}^{\neq 0}}\hat{A}\big(\Omega^{(Y_\gamma\times\mathbb{R}^{\neq 0})/\mathbb{R}^{\neq 0}}\big)\right]_{(1)}. \tag{15}$$

Now (14) implies that

$$\lim_{\epsilon\to 0}\hat{A}\big(\Omega^{(Y_\gamma\times\mathbb{R}^{\neq 0})/\mathbb{R}^{\neq 0}}\big) = \hat{A}(\Omega^{Y_\gamma/I}). \tag{16}$$

One should understand this as a limit of sections of a bundle on Y_γ whose fibers are forms on $Y_\gamma\times\{0\}$. In other words, they are forms on Y_γ with a '$d\epsilon$' term as well. Formula (16) implies that there is no $d\epsilon$ term in the limit, and so the integral over the fibers in (15) vanishes. Therefore, $\lim\limits_{\epsilon\to 0}\frac{d}{d\epsilon}\tau_{Y_\gamma}(\epsilon) = 0$ and so $\text{a-lim}\,\tau_{Y_\gamma} = \lim\limits_{\epsilon\to 0}\tau_{Y_\gamma}(\epsilon)$ exists.

A similar argument proves that τ_γ is invariant under reparametrization. Let $\mathcal{D}$ denote the space of diffeomorphisms $\phi\colon[0,1]\to[0,1]$ with $\phi(0)=0$ and $\phi(1)=1$. We pull back $\pi\colon Y\to Z$ via the map

$$\begin{aligned}[0,1]\times\mathbb{R}^{\neq 0}\times\mathcal{D} &\longrightarrow Z\\ \langle t,\epsilon,\phi\rangle &\longmapsto \gamma\big(\phi(t)\big)\end{aligned}$$

[4] And we correct a mistake in the exposition there.

to construct the family of manifolds

$$\mathcal{Y} \longrightarrow \mathbb{R}^{\neq 0} \times \mathcal{D},$$

where the metric on the fiber over $\langle \epsilon, \phi\rangle$ is (11). As in the previous argument we compute the differential of $\tau_{Y_{\gamma\circ\phi}}(\epsilon,\phi)$ in the adiabatic limit:

$$\lim_{\epsilon\to 0} d\tau_{\langle\epsilon,\phi\rangle} = 2\pi i\ \ \sigma^*\left[\int_{Y/Z} \hat{A}(\Omega^{Y/Z})\right]_{(2)}, \tag{17}$$

where

$$\begin{aligned} \sigma\colon [0,1]\times\mathcal{D} &\longrightarrow Z \\ \langle t,\phi\rangle &\longmapsto \gamma\bigl(\phi(t)\bigr) \end{aligned}$$

We conclude that (17) vanishes since the image of σ is one dimensional—the pullback of a 2-form vanishes.

Lemma 18. *The maps τ_γ are the parallel transport of a connection ∇' on $L\to Z$.*

Remark. Since τ_γ is a unitary transformation ($|\tau_\gamma|=1$), the connection ∇' is also unitary.

Proof. By a general result [F2, Appendix B] it suffices to show that the fiducial parallel transport τ_γ is invariant under reparametrization and composes under gluing. The first statement is contained in the previous lemma. For the second, if γ_1,γ_2 are paths with $\gamma_2(0)=\gamma_1(1)$, then we can compose to get a path $\gamma=\gamma_2\circ\gamma_1$. The gluing law Theorem 8 then implies $\tau_\gamma=\tau_{\gamma_2}\circ\tau_{\gamma_1}$ as required. (Theorem 8 applies to a fixed metric and then we take the adiabatic limit.)

Remark. It is instructive to see in detail how the sign works in this application of the gluing law. Here we cut Y_γ along $Y=Y_{\gamma_2(0)}=Y_{\gamma_1(1)}$ to obtain $Y_\gamma^{\mathrm{cut}}=Y_{\gamma_1}\sqcup Y_{\gamma_2}$. So

$$\begin{aligned} \tau_{\gamma_1} &\in \operatorname{Hom}(L_{\gamma_1(0)},L_{\gamma_1(1)}) \cong \quad L_Y \quad \otimes L^{-1}_{\gamma_1(0)}, \\ \tau_{\gamma_2} &\in \operatorname{Hom}(L_{\gamma_2(0)},L_{\gamma_2(1)}) \cong L_{\gamma_2(1)} \otimes L_Y^{-1}, \end{aligned}$$

where we write $L_Y=L_{\gamma_1(1)}=L_{\gamma_2(0)}$. Thus

$$\tau_{Y_\gamma^{\mathrm{cut}}} = \tau_{\gamma_2}\otimes\tau_{\gamma_1} \in L_{\gamma_2(1)}\otimes L_Y^{-1}\otimes L_Y\otimes L^{-1}_{\gamma_1(0)}. \tag{19}$$

The key point is that the factors are in a different order than in (6) and (7)—now the factor L_Y^{-1} *precedes* the factor L_Y. So the contraction is the usual trace. Put differently, to move (19) to the standard form (6) we introduce a factor of $(-1)^{\operatorname{index} D_Y}$ and this is cancelled by the factor $(-1)^{\operatorname{index} D_Y}$ in the supertrace (9). The upshot is that in this situation the right hand side of (9) is $\tau_{\gamma_2}\circ\tau_{\gamma_1}$ as desired.

It is quite easy to prove from the variation formula Theorem 4 that this new connection agrees with the canonical connection ∇.

Proposition 20. $\nabla' = \nabla$.

Proof. We must show that the parallel transports agree. Let $\gamma\colon [0,1] \to Z$ be a path and fix an element $\ell_0 \in L_{\gamma(0)}$ of unit norm. Then if $\gamma\colon [0,t] \to Z$, $0 \le t \le 1$, is the restriction of γ, and $\tau_t\colon L_{\gamma(0)} \to L_{\gamma(t)}$ the parallel transport of ∇', by definition the path $\ell_t = \tau_t(\ell_0)$ is parallel for ∇'. It suffices to show that $\dfrac{D\tau_t}{Dt} = 0$, where $\dfrac{D}{Dt} = \nabla$ along the path γ. For then $\dfrac{D\tau_t(\ell_0)}{Dt} = 0$ as well, since ℓ_0 is a constant.

Define $T = \{\langle t,s\rangle \in [0,1]\times[0,1] : s \le t\}$ with projection

$$\begin{aligned} \rho\colon T &\longrightarrow [0,1] = I \\ \langle t,s\rangle &\longmapsto t \end{aligned}$$

and a map

$$\begin{aligned} \Gamma\colon T &\longrightarrow Z \\ \langle t,s\rangle &\longmapsto \gamma(s). \end{aligned}$$

Then the pullback $\pi\colon \Gamma^*Y \to T$ determines a family of manifolds $\rho\circ\pi\colon \Gamma^*Y \to [0,1]$ parametrized by $I = [0,1]$. We use the flat metric on T and make $\pi\colon \Gamma^*Y \to T$ a Riemannian submersion. The variation formula Theorem 4 implies

$$\frac{D\tau_t}{Dt} = 2\pi i \int_{\Gamma^*Y/I} \text{a-lim}\left[\hat{A}(\Omega^{\Gamma^*Y/I})\right]_{(1)}. \tag{21}$$

Even before taking the adiabatic limit, the fact that Γ factors through the projection $\langle t,s\rangle \mapsto s$ implies that the right hand side of (21) vanishes.

In view of Proposition 20, to compute the curvature and holonomy of ∇ it suffices to compute the curvature and holonomy of ∇'. Notice that since $L = \mathrm{Det}_{Y/Z}^{-1}$ is the *inverse* determinant line bundle our formulas here have opposite signs to those for $\mathrm{Det}_{Y/Z}$ computed in [BF1], [BF2]. The holonomy is computed from the parallel transport by a straightforward application of the gluing law. We must only be careful about the spin structure. Recall that S^1 has two spin structures. The *nonbounding* spin structure is the trivial double cover of the circle; the *bounding* spin structure is the nontrivial double cover.

Theorem 22 [BF2,Theorem 3.18]**.** *Suppose $\gamma\colon [0,1] \to Z$ is a closed path.*[5] *There is an induced manifold $\hat{Y}_\gamma \to S^1$ obtained by gluing the ends of Y_γ. Then the holonomy of L around γ is*

$$\mathrm{hol}_L(\gamma) = \begin{cases} (-1)^{\operatorname{index} D_Y}\ \text{a-lim}\,\tau_{\hat{Y}_\gamma}, & \textit{nonbounding spin structure on } S^1; \\ \text{a-lim}\,\tau_{\hat{Y}_\gamma}, & \textit{bounding spin structure on } S^1. \end{cases} \tag{23}$$

[5] Recall that we require that $\gamma\big([0,\delta]\big)$ and $\gamma\big([1-\delta,1]\big)$ be constant for some δ.

Here the spin structure on S^1 combines with the spin structure on $T(\hat{Y}_\gamma/S^1)$ to give a spin structure on $\hat{Y}_\gamma$.

Proof. This follows directly from the definition (13) of parallel transport and the gluing law applied to $X = \hat{Y}_\gamma$ and $X^{\text{cut}} = Y_\gamma$. Take first the nonbounding spin structure on S^1, lifted to a spin structure on $\hat{Y}_\gamma$. The induced spin structure on the cut manifold Y_γ is the standard one, with the ends each identified with Y_z, where $z = \gamma(0) = \gamma(1)$. Now for each ϵ the τ-invariant of Y_γ is an element

$$\tau_{Y_\gamma(\epsilon)} \in L_z \otimes L_z^{-1}.$$

Then Theorem 8 implies

$$\tau_{\hat{Y}_\gamma(\epsilon)} = (-1)^{\text{index}\, D_Y} \tau_{Y_\gamma(\epsilon)},$$

where on the right hand side we identify $L_z \otimes L_z^{-1}$ with $\mathbb{C}$ using the *usual contraction*. Now the first equation in (23) follows from the definition of holonomy in terms of parallel transport. To obtain the second equation, consider the identity map of Y_z lifted to the *nontrivial* deck transformation on the spin bundle of Y_z. It induces multiplication by $(-1)^{\text{index}\, D_Y}$ on the inverse determinant line L_z. Apply this transformation to Y_γ before gluing in order to switch spin structures on $\hat{Y}_\gamma$. Then the second equation in (23) follows from the first.

Theorem 24 [BF2,Theorem 1.21]**.** *The curvature Ω^L of the inverse determinant line bundle $L \to Z$ is*

$$\Omega^L = -2\pi i \left[\int_{Y/Z} \hat{A}(\Omega^{Y/Z}) \right]_{(2)} . \tag{25}$$

Proof. For any line bundle we can determine the curvature once we know the holonomy as follows. Suppose $\Gamma\colon D \to Z$ is a map of a disk into Z with boundary map γ. Let $Y_\Gamma = \Gamma^* Y \to D$ be the pullback manifold; then $\partial Y_\Gamma = \hat{Y}_\gamma$. In the following calculation we use the bounding spin structure on S^1 and the induced spin structure on $\hat{Y}_\gamma$.

$$\begin{aligned}
\int_D \Omega^L &= -\log \mathrm{hol}_L(\gamma), \\
&= \text{a-lim}\big(-\log \tau_{\hat{Y}_\gamma}\big), \\
&= \text{a-lim}\left\{ -2\pi i \int_{Y_\Gamma} \hat{A}(\Omega^{Y_\Gamma}) \right\}, \\
&= \int_D (-2\pi i) \int_{Y_\Gamma/D} \hat{A}(\Omega^{Y_\Gamma/D}), \\
&= \int_D \Gamma^* \left\{ -2\pi i \left[\int_{Y/Z} \hat{A}(\Omega^{Y/Z}) \right]_{(2)} \right\}.
\end{aligned} \tag{26}$$

In the fourth line we apply (14). In the third line we apply the index theorem of Atiyah-Patodi-Singer [APS] which asserts that

$$\int_{Y_\Gamma} \hat{A}(\Omega^{Y_\Gamma}) - \frac{\eta_{Y_\Gamma}(0) + \dim \operatorname{Ker} D_{Y_\Gamma}}{2}$$

is a certain index, so in particular is an integer. When Γ shrinks the disk to a point both sides of (26) vanish, so we have chosen the correct logarithm on the right hand side of (26). Since (26) holds for all $\Gamma\colon D \to Z$, equation (25) follows.

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Department of Mathematics, University of Texas, Austin, TX 78712
E-mail address: `dafr@math.utexas.edu`